Problem 25) a) $\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{1}{4}\int_0^{2\pi} \sin^4\theta \, d\theta = \frac{1}{4}\oint_{\text{Unit-Circle}} \left[\frac{(z-z^{-1})}{2i}\right]^4 \frac{dz}{iz}$

$$= \frac{1}{64i}\oint \frac{(z^2-1)^4}{z^5}dz = \frac{1}{64i}\oint \frac{\sum_{m=0}^{4}\binom{4}{m} z^{2m}(-1)^{4-2m}}{z^5}dz$$

Aside from the Term Corresponding to $m=2$, all the Terms integrate to Zero. We will have:

$$\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{1}{64i}\binom{4}{2}\oint \frac{dz}{z} = \frac{2\pi i}{64i}\,\frac{4!}{2!\,2!} = \frac{3\pi}{16}.$$

The same method works for $\int_0^{\pi/2} \cos^4\theta \, d\theta$, yielding the same result.

b) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta = \oint_{\text{Unit-Circle}} \frac{[(z-z^{-1})/2i]^2}{5+2(z+z^{-1})}\frac{dz}{iz} = \frac{i}{8}\oint \frac{(z^2-1)^2}{z^2(z^2+\frac{5}{2}z+1)}dz$

$$= \frac{i}{8}\oint_{\text{Unit-Circle}} \frac{(z^2-1)^2}{z^2(z+\frac{1}{2})(z+2)}dz$$

Inside the Unit Circle then, there is a simple pole at $z_0=-\frac{1}{2}$, and a 2nd-order pole at $z_1=0$.

Residue at $z_0=-\frac{1}{2}$: $\frac{(z_0^2-1)^2}{z_0^2(z_0+2)} = \frac{9/16}{(1/4)(3/2)} = \frac{3}{2}$. ✓

Residue at $z_1=0$: $\left(\frac{(z^2-1)^2}{z^2+\frac{5}{2}z+1}\right)'\Big|_{z=0} = \frac{4z(z^2-1)(z^2+\frac{5}{2}z+1)-(2z+\frac{5}{2})(z^2-1)^2}{(z^2+\frac{5}{2}z+1)^2}\Big|_{z=0}$

$$= \frac{-5/2}{1} = -\frac{5}{2}. \checkmark$$

Consequently, $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta = \frac{i}{8}(2\pi i)\left(\frac{3}{2}-\frac{5}{2}\right) = \frac{\pi}{4}.$ ✓

c) $\int_0^{2\pi} \frac{d\theta}{a^2+\sin^2\theta} = 2\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = 2\int_0^{\pi} \frac{d\theta}{a^2+\frac{1-\cos 2\theta}{2}} = 2\int_0^{2\pi} \frac{d\theta'}{(2a^2+1)-\cos\theta'}$

$\theta' = 2\theta$

This is now the same as Problem 29, with the final answer being:

$$\int_0^{2\pi} \frac{d\theta}{a^2+\sin^2\theta}\, d\theta = 2\,\frac{2\pi}{\sqrt{(2a^2+1)^2-1}} = \frac{4\pi}{\sqrt{4a^4+4a^2}} = \frac{2\pi}{a\sqrt{a^2+1}}.$$

The same procedure works for $\int_0^{2\pi} \frac{d\theta}{a^2+\cos^2\theta}$, and the final answer is the same.

Without the change of variable from θ to 2θ, the problem is harder to solve, as can be seen below.

$$\int_0^{2\pi} \frac{d\theta}{a^2+\sin^2\theta} = \oint_{\substack{\text{unit}\\\text{circle}}} \frac{\frac{dz}{iz}}{a^2+\left(\frac{z-1/z}{2i}\right)^2} = 4i\oint \frac{z\,dz}{-4a^2z^2+(z^2-1)^2}$$

The poles are found from: $(z^2-1)^2 = 4a^2z^2 \Rightarrow z^2-1 = \pm 2az \Rightarrow$

$z^2 \pm 2az - 1 = 0 \Rightarrow z = \mp a \pm \sqrt{a^2+1} \Rightarrow z_1 = -a+\sqrt{a^2+1}$, $z_2 = -a-\sqrt{a^2+1}$,

$z_3 = a+\sqrt{a^2+1}$, $z_4 = a-\sqrt{a^2+1}$. Of these z_1 and z_4 are inside the unit-circle.

$$\int_0^{2\pi} \frac{d\theta}{a^2+\sin^2\theta} = 4i\oint_{\substack{\text{unit}\\\text{circle}}} \frac{z\,dz}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} = 4i\,(2\pi i)\,(\text{Residue at } z_1 + \text{Residue at } z_4)$$

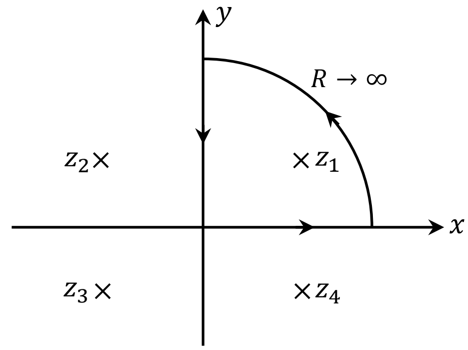
$$= -8\pi\left(\frac{z_1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{z_4}{(z_4-z_1)(z_4-z_2)(z_4-z_3)}\right)$$

$$= -8\pi\left(\frac{-a+\sqrt{a^2+1}}{2\sqrt{a^2+1}\,(-2a)(-2a+2\sqrt{a^2+1})} + \frac{a-\sqrt{a^2+1}}{(2a-2\sqrt{a^2+1})(2a)(-2\sqrt{a^2+1})}\right)$$

$$= -8\pi\left(\frac{1}{-8a\sqrt{a^2+1}} + \frac{1}{-8a\sqrt{a^2+1}}\right) = \frac{2\pi}{a\sqrt{a^2+1}}.$$

Problem 25)

d) To evaluate $\int_0^\infty \frac{\mathrm{d}x}{x^4+4a^4}$, use the contour shown. The poles of the integrand are readily found, as follows:



$$z^4 + 4a^4 = 0 \quad \to \quad z^4 = -4a^4 \quad \to \quad z^2 = \pm \mathrm{i}2a^2$$

$$\to \quad z^2 = 2a^2 e^{\pm \mathrm{i}\pi/2} \quad \to \quad z = \pm\sqrt{2}ae^{\pm\mathrm{i}\pi/4}.$$

The only pole that is inside the contour is $z_1 = \sqrt{2}ae^{\mathrm{i}\pi/4}$.

$$\text{Residue at } z_1 = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$= \frac{1}{2a \times 2\sqrt{2}a\exp(\mathrm{i}\pi/4) \times 2a\mathrm{i}} = \frac{1}{\mathrm{i}8\sqrt{2}a^3\exp(\mathrm{i}\pi/4)}.$$

Loop integral when $R \to \infty$:

$$\int_0^\infty \frac{\mathrm{d}x}{x^4+4a^4} - \int_0^\infty \frac{\mathrm{i}\mathrm{d}y}{(\mathrm{i}y)^4+4a^4} = (1-\mathrm{i})\int_0^\infty \frac{\mathrm{d}x}{x^4+4a^4} = \frac{2\pi\mathrm{i}}{\mathrm{i}8\sqrt{2}a^3\exp(\mathrm{i}\pi/4)}$$

$$\to \quad \int_0^\infty \frac{\mathrm{d}x}{x^4+4a^4} = \frac{\pi}{4\sqrt{2}a^3\exp(\mathrm{i}\pi/4)(1-\mathrm{i})} = \frac{\pi}{8a^3}.$$

e) To evaluate $\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2}$, use an infinitely large semi-circular contour in the upper half of the complex plane. The poles of the integrand are readily found by setting $z^2 + a^2 = 0$, which yields $z_{1,2} = \pm\mathrm{i}a$. The pole inside the contour is $z_1 = \mathrm{i}a$, and it is a second-order pole.

$$\text{Residue at } z = \mathrm{i}a\text{:} \quad \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{(z+\mathrm{i}a)^2}\right]_{z\,=\,\mathrm{i}a} = -2(z+\mathrm{i}a)^{-3}\big|_{z\,=\,\mathrm{i}a} = -2(2\mathrm{i}a)^{-3} = \frac{1}{\mathrm{i}4a^3}.$$

Therefore,

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{1}{2}(2\pi\mathrm{i})\left(\frac{1}{\mathrm{i}4a^3}\right) = \frac{\pi}{4a^3}.$$